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Univalence of a Function f and Its Successive Derivatives When f Satisfies a Differential Equation, II

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Solutions of second order differential equations with quadratic polynomials as coefficients are studied. By a suitable choice of constants it is shown that there exists a solution which is an entire function of exponential type one and such that this function and all its derivatives are close-to-convex in the unit disc. By another choice of constants we get an entire solution of exponential type $\sqrt{2}$ with gap power series such that all odd, or all even, derivatives are close-to-convex in the unit disc. © 1989 Academic Press, Inc.

1. In this paper we consider extensions of the following theorem of R. P. Boas [1].

THEOREM A [1]. *If $f(z)$ is an entire function of exponential type less than* $\log 2$, *and at most a finite number of its derivatives are univalent in $D = \{z, |z| < 1\}$, then $f(z)$ is a polynomial.*

An extension to even or odd entire functions is given by G. A. Read [3] and to entire functions with some derivatives univalent by S. M. Shah and S. Y. Trimble [5–7], and S. M. Shah [8].

For theorems of converse type, where an infinite number of derivatives of f being univalent in D implies that f must be entire, see [4, 7]. We extend some results proved in [8].

We suppose in what follows that the constant coefficients of the differential equation (DE) (1.1) are real, and β_1 is neither an integer nor zero.

THEOREM 1. *Consider the DE*

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)\, w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)\, w = 0. \tag{1.1}$$

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(i) *Suppose* $\gamma_2 = 0$. *Then there exists an entire solution* f:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 = 1. \tag{1.2}$$

If further

$$\beta_0 \leqslant 0, \gamma_1 < 0, \beta_1 > 0, \gamma_0 = 0, |\beta_0| \leqslant 1, \gamma_2 = 0, |\gamma_1| \leqslant \beta_1, \tag{1.3}$$

then $f, f', f'', \ldots$ *are all close-to-convex in* D *and* $\log M(r, f) \sim |\beta_0|\, r$.

(ii) *Set* $\beta_1 + \gamma_2 = 0$ *in DE* (1.1). *Then there exists an entire solution* F:

$$F(z) = z + \sum_{2}^{\infty} a_n z^n. \tag{1.4}$$

Suppose further

$$\beta_0 \leqslant 0, |\beta_0| + |\gamma_1| > 0, \gamma_1 \leqslant 0, |\beta_0| \leqslant 1, \beta_1 > 0, |\gamma_1| \leqslant \beta_1/2. \tag{1.5}$$

Then F *is of exponential type*, $\log M(r, F) \sim |\beta_0|\, r$, *and* $F, F', F'', \ldots$ *are all close-to-convex in* D.

THEOREM 2. (i) *Suppose in the DE* (1.1)

$$\beta_0 = \gamma_1 = \gamma_2 = 0, \beta_1 \geqslant 0, |\gamma_0| \leqslant 2, \gamma_0 < 0. \tag{1.6}$$

Then there exists an even entire solution

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 = 1, a_1 = 0.$$

Further all odd derivatives $f', f''', \ldots$ *are close-to-convex in* D *and* $\log M(r, f) \sim |\gamma_0|^{1/2} r$.

(ii) *Set* $\beta_1 + \gamma_2 = 0$ *in the DE* (1.1). *Then there exists an entire solution* F:

$$F(z) = z + \sum_{2}^{\infty} a_n z^n.$$

Suppose further that

$$\beta_0 = \gamma_1 = 0, \beta_1 \geqslant 0, |\gamma_0| \leqslant 2, \gamma_0 \neq 0. \tag{1.7}$$

Then all even derivatives $F, F'', F^{(iv)}, \ldots$ *are close-to-convex in* D *and* $\log M(r, F) \sim |\gamma_0|^{1/2} r$.

We give in Section 5 some examples where the DE satisfies the conditions of Theorem 1 or of Theorem 2. Some of these conditions can be relaxed.

THEOREM 3. (i) *Suppose in the DE* (1.1)

$$\gamma_0 \neq 0, \gamma_2 = 0 = \beta_0 = \gamma_1, \beta_1 \geqslant 1, |\gamma_0|^{1/2} \leqslant \log(2+\sqrt{3}) = 1.31.... \quad (1.8)$$

Then there exists an entire solution f:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 = 1, a_1 = 0. \quad (1.9)$$

Further all coefficients $a_1, a_3, ...$ *are zero and all odd derivatives* $f', f''', ...$ *are univalent in D. Also f is of perfectly regular growth and of type* $|\gamma_0|^{1/2}$.

(ii) *Suppose in the DE* (1.1)

$$\gamma_0 \neq 0, \beta_1 + \gamma_2 = 0, \beta_0 = \gamma_1 = 0, \beta_1 \geqslant 1, |\gamma_0|^{1/2} \leqslant \log(2+\sqrt{3}). \quad (1.10)$$

Then there exists an entire solution F:

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

F is of perfectly regular growth and of type $|\gamma_0|^{1/2}$, *and F and all even derivatives* $F^{(2k)}$, $k = 1, 2, ...$, *are univalent in D.*

Remark. The conclusions of these theorems hold even if $|\beta_0| = 1$ in (1.3) and (1.5) and $|\gamma_0| = 2$ in (1.6) and (1.7).

2. *Proof of Theorem* 1. (i) We use the Alexander theorem [2, p. 9]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and the coefficients satisfy

$$1 \geqslant 2a_2 \geqslant 3a_3 \geqslant \cdots \geqslant na_n \geqslant (n+1)\, a_{n+1} \geqslant \cdots 0,$$

then $f(z)$ is close-to-convex in D. The same conclusion holds if $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}$ and $1 \geqslant 3a_3 \geqslant 5a_5 \geqslant \cdots (2n-1)\, a_{2n-1} \geqslant \cdots 0$. From (1.2) and (1.3) we get

$$a_n = \frac{|\beta_0|(n-1) + |\gamma_1|}{n(n+\beta_1 - 1)}\, a_{n-1}, \qquad n \geqslant 2. \quad (2.1)$$

Hence $a_0, a_1, ...$ are all positive. Since

$$2a_2 = (\beta_0 + \gamma_1)\frac{\gamma_1}{\beta_1}(1+\beta_1)^{-1} \qquad \text{and} \qquad a_1 = \frac{|\gamma_1|}{\beta_1},$$

a_2 is positive and $2a_2 \leqslant a_1$. Now $a_n/a_{n-1} = (|\beta_0|(n-1)+|\gamma_1|)/n(n+\beta_1-1) \leqslant (n-1)/n$ for $n \geqslant 2$. Hence $(f(z)-f(0))/f'(0) = (f(z)-a_0)/a_1$ is close-to-convex in D, and since $a_0>0$, $a_1>0$ the conclusion for f follows.

For $f^{(k)}(z)$, consider

$$\frac{f^{(k)}(z)-f^{(k)}(0)}{f^{(k+1)}(0)} = z + \frac{z^2}{2!}\frac{f^{(k+2)}(0)}{f^{(k+1)}(0)} + \cdots + \frac{z^n}{n!}\frac{f^{(n+k)}(0)}{f^{(k+1)}(0)} + \cdots.$$

We prove

$$1 \geqslant \frac{2}{2!}\frac{f^{(k+2)}(0)}{f^{(k+1)}(0)} \geqslant \cdots \geqslant \frac{n}{n!}\frac{f^{(n+k)}(0)}{f^{(k+1)}(0)} \geqslant \cdots 0. \tag{2.2}$$

Now

$$\frac{f^{(k+2)}(0)}{f^{(k+1)}(0)} = \frac{(k+2)!\,a_{k+2}}{(k+1)!\,a_{k+1}} = (k+2)\frac{a_{k+2}}{a_{k+1}} > 0.$$

Since

$$\frac{|\beta_0|(k+1)+|\gamma_1|}{(k+2)(k+1+\beta_1)} \leqslant \frac{1}{k+2},$$

we get $f^{(k+1)}(0) \geqslant f^{(k+2)}(0)$. Next we show

$$\frac{(n-1)}{(n-1)!}\frac{f^{(k+n-1)}(0)}{f^{(k+1)}(0)} \geqslant \frac{n}{n!}\frac{f^{(n+k)}(0)}{f^{(k+1)}(0)},$$

that is,

$$f^{(k+n-1)}(0) \geqslant \frac{1}{n-1} f^{(n+k)}(0)$$

or

$$(k+n-1)!\,a_{k+n-1} \geqslant \frac{(k+n)!\,a_{k+n}}{n-1}.$$

But

$$\frac{a_{k+n}}{a_{k+n-1}} = \frac{|\beta_0|(k+n-1)+|\gamma_1|}{(k+n)(k+n-1+\beta_1)} \leqslant \frac{n-1}{k+n}$$

follows from conditions (1.3). Hence $(f^{(k)}(z)-f^{(k)}(0))/f^{(k+1)}(0)$ is close-to-convex for $k=0, 1, 2, \ldots$, and so $f^{(k)}(z)$ and $f(z)$ and all its derivatives are close-to-convex in D. Further if $\nu(r)$ denotes the index of the maximum term, then from (2.1) we get $\lim_{r\to\infty}(\nu(r)/r) = |\beta_0|$ and $\log M(r, f) \sim |\beta_0|\, r$.

(ii) Let $\beta_1 + \gamma_2 = 0$. Then the indicial equation has one root equal to 1 and so $F(z) = z + \sum_2^\infty a_n z^n$ is an entire solution [8]. Now

$$a_n = \frac{|\beta_0|(n-1) + |\gamma_1|}{(n-1)(n+\beta_1)}\, a_{n-1}, \qquad n \geqslant 2,$$

and so all coefficients are positive. We prove as in (i) that $1 \geqslant 2a_2 \geqslant 3a_3 \geqslant \cdots \geqslant na_n \geqslant \cdots$ and then conclude that F will be close-to-convex in D. For $F^{(k)}(z)$ consider, for $k \geqslant 1$,

$$\frac{F^{(k)}(z) - F^{(k)}(0)}{F^{(k+1)}(0)} = z + \frac{z^2}{2!}\frac{F^{(k+2)}(0)}{F^{(k+1)}(0)} + \frac{z^3}{3!}\frac{F^{(k+3)}(0)}{F^{(k+1)}(0)} + \cdots.$$

We prove first

$$1 \geqslant \frac{2}{2!}\frac{F^{(k+2)}(0)}{F^{(k+1)}(0)} = \frac{a_{k+2}(k+2)!}{a_{k+1}(k+1)!},$$

that is,

$$\frac{a_{k+2}}{a_{k+1}} = \frac{(k+1)\,|\beta_0| + |\gamma_1|}{(k+1)(k+2+\beta_1)} \leqslant \frac{1}{k+2}.$$

This follows since $|\beta_0| < 1$, $|\gamma_1| \leqslant \beta_1/2$. Next we show that

$$\frac{n}{n!}\, F^{(n+k)}(0)/F^{(k+1)}(0) \leqslant \frac{n-1}{(n-1)!}\, F^{(k+n-1)}(0)/F^{(k+1)}(0),$$

that is, $a_{k+n}/a_{k+n-1} \leqslant (n-1)/(k+n)$. Now $a_{k+n}/a_{k+n-1} = \{|\beta_0|(k+n-1) + |\gamma_1|\}/(k+n-1)(k+n+\beta_1)$, and since $|\beta_0| \leqslant 1$ and $|\gamma_1| \leqslant \beta_1/2$, the right hand side of the previous step is less than or equal to $(n-1)/(k+n)$. This shows that $F^{(k)}(z)$, $k = 1, 2, \ldots$ are all close-to-convex in D. Type $(F) = |\beta_0|$.

3. *Proof of Theorem* 2. (i) Let $\gamma_2 = 0$. Then the DE (1.1) has an entire solution $f(z) = \sum_0^\infty a_n z^n$, $a_0 = 1$, $a_1 = 0$ [8]. Let

$$\beta_0 = \gamma_i = 0,\ \gamma_0 < 0,\ |\gamma_0| \leqslant 2 \text{ and } \beta_1 \geqslant 0. \tag{3.1}$$

Then

$$a_n = -\frac{\gamma_0 a_{n-2}}{n(n-1+\beta_1)} = \frac{|\gamma_0|\, a_{n-2}}{n(n-1+\beta_1)}.$$

Hence $a_0, a_2, \ldots, a_{2n}, \ldots$ are all positive and $a_1, a_3, \ldots$ are all zero. So

$$f(z) = 1 + a_2 z^2 + a_4 z^4 + \cdots + a_{2n} z^{2n} + \cdots.$$

f is even and not univalent in D. Now

$$\frac{f'(z)}{2a_2} = z + \frac{4a_4}{2a_2} z^3 + \cdots + \frac{2na_{2n}}{2a_2} z^{2n-1} + \cdots.$$

Since $\beta_1 \geqslant 0$ and $|\gamma_0| \leqslant 2$, we get $\frac{1}{6} \geqslant a_4/a_2$. Consider, for $n \geqslant 3$,

$$\frac{a_{2n}}{a_{2n-2}} = \frac{|\gamma_0|}{2n(2n-1+\beta_1)}. \tag{3.2}$$

Since $\beta_1 \geqslant 0$ and $|\gamma_0| \leqslant 2$ we see that the right side of (3.2) is less than or equal to $(n-1)(2n-3)/n(2n-1)$. Consequently f' is close-to-convex in D. Consider now, for $k > 1$,

$$\begin{aligned} f^{(2k-1)}(z) &= \sum_{p=0}^{\infty} \frac{(p+2k-1)!}{p!} a_{p+2k-1} z^p \\ &= \frac{(2k-1)!}{1!} a_{2k-1} + \frac{(2k)!}{1!} a_{2k} z + \cdots \\ &\quad + \frac{(2n-1+2k-1)!}{(2n-1)!} a_{2n-1+2k-1} z^{2n-1} + \cdots. \end{aligned}$$

Then

$$(2n-1) \frac{(2n+2k-2)!}{(2n-1)!} a_{(2n+2k-2)} \geqslant (2n+1) \frac{(2n+2k)!}{(2n+1)!} a_{2n+2k}. \tag{3.3}$$

Since

$$\frac{a_{2n+2k}}{a_{2n+2k-2}} = \frac{|\gamma_0|}{(2n+2k)\{2n+2k-1+\beta_1\}},$$

the inequality (3.3) follows from the conditions (3.1). This proves that $(f^{(2k-1)}(z) - f^{(2k-1)}(0))/f^{(2k)}(0)$ is close-to-convex in D and the proof of (i) of Theorem 2 is complete.

The proof of (ii) is similar. So we only sketch it. Assume the conditions (1.7). Then there exists an entire solution F:

$$F(z) = z + \sum_{2}^{\infty} a_n z^n.$$

From the DE (1.1), we get

$$a_n = \frac{|\gamma_0|\, a_{n-2}}{(n-1)(n+\beta_1)}. \tag{3.4}$$

Since $a_0 = 0$, we get from (3.4) that $a_2 = a_4 = \cdots = 0$ and that

$$F(z) = z + a_3 z^3 + a_5 z^5 + \cdots.$$

Now $a_3 = |\gamma_0|/2(3+\beta_1)$, and so $3a_3 = 3\,|\gamma_0|/(6+2\beta_1) \leqslant 1$ since $|\gamma_0| \leqslant 2$ and $\beta_1 \geqslant 0$. Similarly $(2n+1)\, a_{2n+1} = (2n+1)(|\gamma_0|\, a_{2n-1}/2n(2n+1+\beta_1)) \leqslant (2n-1)\, a_{2n-1}$. Hence F is close-to-convex in D. F' is even, and so we consider

$$\frac{F''(z) - F''(0)}{F'''(0)} = z + \cdots + \frac{(2n-1)(2n-2)}{3!\, a_3} a_{2n-1} z^{2n-3} + \cdots.$$

Now from (3.4),

$$\frac{(2n-1)(2n-2)(2n-3)}{3!\, a_3} a_{2n-1} \geqslant \frac{(2n+1)(2n)(2n-1)}{3!\, a_3} a_{2n+1}$$

for $n \geqslant 2$. Hence F'' is close-to-convex in D. Similarly $F^{(2k)}$ is close-to-convex in D. Also the type of F is $|\gamma_0|^{1/2}$.

4. *Proof of Theorem* 3. Let (1.9) be an entire solution of the DE (1.1) where the coefficients satisfy conditions (1.8). Then

$$a_n = \frac{-\gamma_0 a_{n-2}}{n(n-1+\beta_1)}. \tag{4.1}$$

Hence $a_1 = a_3 = a_5 = \cdots = 0$ and $f(z) = a_0 + a_2 z^2 + a_4 z^4 + \cdots + a_{2n} z^{2n} + \cdots$, which is even and not univalent.

Consider univalence of $f^{(2n+1)}$. Now

$$\frac{(2n+3)!}{2!} 2\,|a_{(2n+3)}| + \frac{(2n+4)!}{3!} 3\,|a_{(2n+4)}| + \cdots$$

$$\leqslant \frac{|\gamma|_0}{2!} \frac{|a_{2n+2}|(2n+2)!\,(2n+3)(2n+4)}{(2n+4)(2n+3+\beta_1)}$$

$$+ \frac{(2n+1)!}{4!} |\gamma_0|^2\, |a_{2n+2}| \frac{(2n+2)!\,(2n+3)(2n+4)(2n+5)(2n+6)}{(2n+4)(2n+6)(2n+5+\beta_1)(2n+3+\beta_1)} + \cdots$$

$$\leqslant (2n+2)!\,|a_{2n+2}|\left\{\frac{|\gamma_0|}{2!}+\frac{|\gamma_0|^2}{4!}+\cdots\right\}$$

$$=(2n+2)!\,|a_{2n+2}|\left\{\frac{\exp(|\gamma_0|^{1/2})+\exp(-|\gamma_0|^{1/2})-2}{2!}\right\}$$

$$\leqslant (2n+2)!\,|a_{2n+2}| \qquad \text{since} \quad |\gamma_0|^{1/2}\leqslant \log(2+\sqrt{3}).$$

Hence $f^{(2n+1)}(z)$ is univalent in D. Also by considering $\lim_{r\to\infty}(\nu(r)/r)$ we see that f is of perfectly regular growth and of type $|\gamma_0|^{1/2}$. The proof of (ii) is similar and omitted.

5. We now give some examples.

(i) $z^2w''+(-z^2+z)\,w'-w=0$. Here $\beta_1+\gamma_2=0$, $\beta_0=-1$, $\beta_1=1$, $\gamma_0=0=\gamma_1$, and $\gamma_2=-1$. The conditions of Theorem 1(ii) are satisfied. One solution is

$$F(z)=\frac{2}{z}\{e^z-(1+z)\}=z+\frac{2!}{3!}z^2+\cdots.$$

All coefficients are positive, and F and each $F^{(k)}$ are close-to-convex in D. Type$(F)=1$.

(ii) $z^2w''+(-z^2+z)\,w'+\lambda zw=0$. Here $\gamma_2=0$, $\beta_0=-1$, $\beta_1=1$, $\gamma_0=0$, and $\gamma_1=\lambda$. Choose $0>\lambda\geqslant -1$. Then the conditions of Theorem 1(i) are all satisfied. A solution is

$$f(z)=1+\sum_{k=1}^{\infty}\frac{|\lambda|(|\lambda|+1)\cdots(|\lambda|+k-1)}{(k!)^2}z^k.$$

All coefficients are positive, and hence f and each $f^{(k)}$ are close-to-convex in D. Further, type$(f)=1$.

(iii) $z^2w''+zw'-z^2w=0$. Here $\beta_0=0$, $\beta_1=1$, $\gamma_0=-1$, and $\gamma_1=0=\gamma_2$. So all conditions of Theorem 2(i) are satisfied. The solution is

$$f(z)=\sum_{k=0}^{\infty}\frac{z^{2k}}{2^{2k}(k!)^2}.$$

Type of $(f)=|\gamma_0|^{1/2}=1$ and $f', f''', \ldots$ are close-to-convex in D.

(iv) Modified Bessel function of order p: $I_p(w)$. $z^2w''+zw'-(z^2+p^2)\,w=0$. Here $\beta_0=0$, $\beta_1=1$, $\gamma_0=-1$, $\gamma_1=0$, and $\gamma_2=-p^2$. If we take $p=0$, the conditions of Theorem 2(i) are satisfied and the solution is

$$I_0(z)=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{z}{2}\right)^{2n}.$$

I_0 is of type 1, and all odd derivatives $I_0', I_0''', \ldots$ are close-to-convex in D. Take $p=1$. Then the conditions of Theorem 2(ii) are satisfied and

$$I_1(z)=\left(\frac{z}{2}\right)\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}\left(\frac{z}{2}\right)^{2n}.$$

Here $F(z)=2I_1(z)$.

(v) Bessel function of order p. $z^2w''+zw'+(z^2-p^2)w=0$. Take $p=0$. Then $\gamma_2=0=\beta_0=\gamma_1$, $\beta_1=1$, and $\gamma_0=1$. The conditions of Theorem 3(i) are satisfied and all odd derivatives of $J_0(z)$ are univalent in D. Take $p=1$. Then $\beta_0=0$, $\beta_1=1$, $\gamma_0=1$, $\gamma_1=0$, and $\gamma_2=-1$, the conditions of Theorem 3(ii) are satisfied, $F(z)=2J_1(z)$, and F and all even derivatives, $F^{(2k)}$, $k=1, 2, \ldots$, are univalent in D.

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